Topology and its Applications 159 (2012) 3340–3354

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Stellar discriminants and equipartitions

Dmitry N. Kozlov

Department of Mathematics, University of Bremen, 28334 Bremen, Federal Republic of Germany

ARTICLE INFO

Article history:
Received 28 April 2011
Received in revised form 25 July 2012
Accepted 25 July 2012

MSC:
primary 57Q65
secondary 68M10

Keywords:
Stellar subdivisions
Barycentric subdivisions
Combinatorial algebraic topology

ABSTRACT

In this paper we study the following problem: given a geometric d-simplex Δ and the set S of n points in the interior of Δ, find a stellar subdivision of Δ, such that the interiors of all the d-simplices of that subdivision contain equally many points from S.

We introduce the relevant for this problem notion of points being in general position, and give a precise geometric definition of the corresponding stellar discriminant. We show that if points of S are in general position, then such a stellar subdivision always exists, and present an algorithm to find its center using quadratic (in n) time. If the requirement of being in general position is dropped, this is no longer the case. We give an example where the minimal gap in the distribution of points in any stellar subdivision is linear in n.

We then apply our result to a variety of contexts, specifically: fast barycentric embeddings of geometric simplicial complexes, equipartition problems in tropical geometry, and maintaining a balanced system of master sensors in a sensor network.

1. Introduction

The starting point of the investigation undertaken in this paper was the following result from PL-topology.

Theorem 1.1. *([4, Theorem I.2]) Given two finite geometric simplicial complexes K and L in $\mathbb{R}^d$, such that $|L| \subseteq |K|$, there exist a positive integer n and a subcomplex $\widetilde{L}$ of $\mathrm{bd}^n K$, such that $\widetilde{L}$ subdivides the complex L.*

Here bd denotes the generalized barycentric subdivision, where we are allowed to pick an arbitrary barycenter on any simplex, see [4, Chapter 1, §B], and Definition 2.3 below. A natural question which arises in this context is the following:

Determine the minimal possible number of subdivisions $\eta(K, L)$ for each given pair of geometric simplicial complexes K and L.

To answer this question can be hard even for seemingly elementary situations. Consider for example the case when K is a d-simplex, and L is the chromatic subdivision of a simplex which was introduced by Herlihy and Shavit, see [5]. We have found that $\eta(K, L) = 2$ for the case $d = 2$, see Fig. 1.1. In the general case, all that is known is that $2 \leqslant \eta(K, L) \leqslant d + 1$, where the upper bound is due to Gafni, see [3].

Returning to the general case, we notice that, somewhat curiously, already the case $\dim L = 0$, $\dim K = 2$ is an interesting and not completely understood case. Our approach in this paper is to consider stellar subdivisions first. We shall prove in Theorem 3.5 that when Δ is a d-simplex, and S is a set of n points in general position, then there always exists a stellar subdivision of Δ, such that each of the $d+1$ top-dimensional simplices contains precisely $\lfloor \frac{n}{d+1} \rfloor$ points from S in its interior.

This is the key result, from which a number of corollaries in various contexts will follow. First, we shall get an asymptotic result for the number of barycentric subdivisions needed to embed a 0-dimensional complex into a d-dimensional one.

E-mail address: dfk@math.uni-bremen.de.

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.07.014

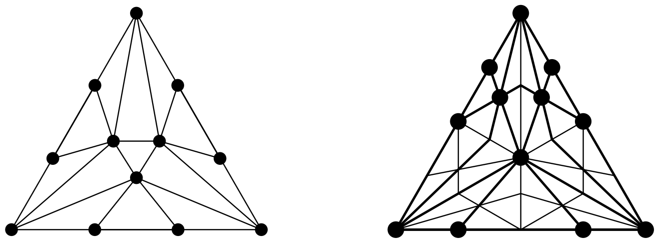

Fig. 1.1. The chromatic subdivision of a 2-simplex and its embedding in the second barycentric subdivision.

Secondly, letting the simplex grow infinitely large, our results will imply similar results in tropical geometry, pertaining to equipartitions of sets of points by a tropical hyperplane. Finally, we shall also present an application to maintaining a balanced system of master sensors in a sensor network.

We call the locus of the point configurations which are not in general position with respect to the stellar subdivision the *stellar discriminant*. One of the main challenges for the subsequent research could be to describe and analyze the stellar discriminant from the point of view of arrangement theory, see [9].

2. Preliminaries

2.1. Geometric simplicial complexes

For an arbitrary nonempty set $A \subseteq \mathbb{R}^n$, let $\operatorname{conv}(A)$ denote the convex hull of A; we set $\operatorname{conv}(\emptyset) := \emptyset$. For a nonnegative integer d, an *embedded d-simplex* in $\mathbb{R}^n$ is a convex hull of $d+1$ affinely independent points in $\mathbb{R}^n$. Given such $d+1$ affinely independent points $v_1, \ldots, v_{d+1}$, we say that $x = \lambda_1 v_1 + \cdots + \lambda_{d+1} v_{d+1}$ is a *convex linear combination* of $v_1, \ldots, v_{d+1}$, whenever $\lambda_1, \ldots, \lambda_{d+1}$ are nonnegative real numbers satisfying $\lambda_1 + \cdots + \lambda_{d+1} = 1$. All convex linear combinations of $v_1, \ldots, v_{d+1}$ form the convex hull of $v_1, \ldots, v_{d+1}$, which is the d-simplex spanned by $v_1, \ldots, v_{d+1}$. Since all the convex linear combinations are distinct, each point in that simplex determines the numbers $\lambda_1, \ldots, \lambda_{d+1}$ uniquely. These are called *barycentric coordinates* of x with respect to $v_1, \ldots, v_{d+1}$.

Given an embedded d-simplex $\Delta = \operatorname{conv}(v_1, \ldots, v_{d+1})$, we set $V(\Delta) := \{v_1, \ldots, v_{d+1}\}$. The *interior* of Δ, denoted by $\operatorname{int}\Delta$, consists of all the points whose barycentric coordinates are all nonzero; in particular, for $d = 0$ we get $\operatorname{int}\Delta = v_1$, and for $d = n-1$ the interior $\operatorname{int}\Delta$ coincides with the topological interior of Δ. The *boundary* of Δ, denoted $\partial\Delta$, is taken to be $\Delta \setminus \operatorname{int}\Delta$. The *boundary simplices* of Δ are all $\operatorname{conv}(\{v_i\}_{i \in I})$, for $I \subset [d+1]$, including $I = \emptyset$. Let $\mathcal{F}(\Delta)$ denote the set of embedded simplices consisting of Δ together with all its boundary simplices.

A *geometric simplicial complex* K is a finite collection of embedded simplices in $\mathbb{R}^n$, such that the following conditions are satisfied:

(1) if $\sigma \in K$, and δ is a boundary simplex of σ, then $\delta \in K$;
(2) if $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2 \in \mathcal{F}(\sigma_1) \cap \mathcal{F}(\sigma_2)$.

When K is a geometric simplicial complex we shall often skip the word “embedded” when referring to the simplices from $\mathcal{F}(\Delta)$. Let $|K| \subset \mathbb{R}^n$ denote the union of all simplices in K and call $|K|$ the *geometric realization* of K. In particular, for an embedded d-simplex Δ, we have $|\mathcal{F}(\Delta)| = \Delta$. For every $x \in |K|$ there exists a unique $\sigma \in K$, such that $x \in \operatorname{int}\sigma$, we call this simplex $\sigma(K, x)$, or simply $\sigma(x)$.

Definition 2.1. Assume that K is a geometric simplicial complex, and $\sigma \in K$.

- The open star of σ, denoted $\operatorname{ostar}(K,\sigma)$, is defined by
$$\operatorname{ostar}(K,\sigma) := \{\tau \in K \mid \sigma \in \mathcal{F}(\tau)\}.$$
- The closed star of σ, denoted $\operatorname{star}(K,\sigma)$, is defined by
$$\operatorname{star}(K,\sigma) := \bigcup_{\tau \in \operatorname{ostar}(K,\sigma)} \mathcal{F}(\tau).$$
- The link of σ, denoted $\operatorname{link}(K,\sigma)$, is defined by
$$\operatorname{link}(K,\sigma) := \{\tau \in K \mid \tau \in \operatorname{star}(K,\sigma),\ \tau \cap \sigma = \emptyset\}.$$

When we just say the star of σ we shall always mean the closed star. Clearly, $\operatorname{star}(K,\sigma)$ and $\operatorname{link}(K,\sigma)$ are again geometric simplicial complexes.

For future reference, recall that a *polyhedron* in $\mathbb{R}^n$ is a solution set of a finite system of linear equalities and inequalities in n variables. Given a polyhedron, a nonempty polyhedron $Q \neq P$ is called a *boundary polyhedron* of P if there exists an open halfspace H such that $H \cap P = \emptyset$ and $\overline{H} \cap P = Q$, where $\overline{H}$ denotes the closure of H. A *polyhedral complex* is a collection of polyhedra Σ, which is closed under taking intersections, and such that whenever $P \in \Sigma$ and Q is a boundary polyhedron of P, we also have $Q \in \Sigma$. *Dimension* of a polyhedron P is the minimal possible dimension of a linear subspace containing P; dimension of a polyhedral complex is the maximal dimension of its member polyhedra. We use the convention $\dim \emptyset = -\infty$. Given a polyhedral complex Σ, its *geometric realization* $|\Sigma|$ is the union of the polyhedra from Σ. If $|\Sigma_1| = |\Sigma_2|$ for polyhedral complexes Σ_1 and Σ_2, then $\dim \Sigma_1 = \dim \Sigma_2$. Finally, given two polyhedral complexes Σ_1 and Σ_2, we set $\Sigma_1 \cap \Sigma_2 := \{P_1 \cap P_2 \mid P_1 \in \Sigma_1,\ P_2 \in \Sigma_2\}$. Clearly, $\Sigma_1 \cap \Sigma_2$ is again a polyhedral complex and we have $|\Sigma_1 \cap \Sigma_2| = |\Sigma_1| \cap |\Sigma_2|$.

2.2. Subdivisions

Given two geometric simplicial complexes K and L, we say that L is a *subdivision* of K if $|K| = |L|$ and every simplex of L is contained in some simplex of K. We now define a few specific subdivisions.

Definition 2.2. Let K be a geometric simplicial complex, and $a \in |K|$. The **stellar subdivision of** K **centered at** a, denoted by $\mathrm{sd}(K, a)$, is the geometric simplicial complex obtained from K by replacing the open star of $\sigma(a)$ with the cone whose apex is a, and whose base is the link of $\sigma(a)$.

It follows immediately from the definition that $\mathrm{sd}(K, a)$ is a subdivision of K in the sense above. When taking the stellar subdivision the face lattice of K changes, and the change depends only on the simplex $\sigma(a)$, not on the choice of the point a. We refer the reader to [1,2] for the precise description as well as connections to combinatorial blowups.

Assume Δ is an embedded d-simplex whose vertices are ordered, say $V(\Delta) = \{e_1, \dots, e_{d+1}\}$, and assume $a \in \operatorname{int} \Delta$. Let $\Delta_i(a)$ denote the d-simplex in $\mathrm{sd}(\mathcal{F}(\Delta), a)$ whose set of vertices is $\{a\} \cup V(\Delta) \setminus \{e_i\}$. Accordingly, the simplices of $\mathrm{sd}(\mathcal{F}(\Delta), a)$ whose interiors cover the interior of Δ are indexed by nonempty subsets of $[d+1]$, namely, we set $\Delta_I(a) := \bigcap_{i \in I} \Delta_i(a)$, for all $\emptyset \neq I \subseteq [d+1]$, in particular, $\Delta_{\{i\}}(a) = \Delta_i(a)$, for all $i \in [d+1]$, and $\Delta_{[d+1]} = \{a\}$. When the center of the stellar subdivision is clear, we shall omit its mentioning and simply write Δ_i and Δ_I.

Similarly to the stellar case, we allow an arbitrary choice of subdivision centers in the definition of the barycentric subdivision.

Definition 2.3. Assume that we have is a geometric simplicial complex K, and a function $f : K \setminus \{\emptyset\} \to |K|$, such that $f(\Delta) \in \operatorname{int} \Delta$, for all $\Delta \in K \setminus \{\emptyset\}$. The **barycentric subdivision of** K **with respect to** f, denoted $\mathrm{bd}(K, f)$, is the geometric simplicial complex whose simplices are convex hulls of sets $\{f(\delta_1), \dots, f(\delta_t)\}$, ranging over all possible chains of simplices $\delta_1 \subset \cdots \subset \delta_t$, where $\delta_i \in K$, for all $i \in [t]$.

Again, for any function f satisfying conditions of Definition 2.3, the geometric simplicial complex $\mathrm{bd}(K, f)$ is a subdivision of K in the sense above.

It is easy to see that $\mathrm{bd}(K, f)$ can be obtained by the following sequence of stellar subdivisions: let d be the dimension of K, perform the stellar subdivisions of all d-simplices of K picking the values of f as corresponding centers, then perform the stellar subdivisions of all $(d-1)$-simplices of K, and so on, until the dimension 1 is reached. Alternatively, one could take any inverse linear extension of the face poset of K and perform stellar subdivision along that extension, taking values of f as corresponding centers.

We refer the reader to the author's monograph [7], for further concepts of combinatorial topology used in the rest of the paper and proceed now with the more technical details.

2.3. Barycentric coordinate representation of stellar subdivisions

Consider an embedded d-simplex Δ. Assume that points x and a belong to the interior of Δ. We would like to describe, in terms of the barycentric coordinates of x and a, which of the simplices of $\mathrm{sd}(\mathcal{F}(\Delta), a)$ contains the point x in its interior. Without loss of generality we may assume that Δ is the *standard simplex* in $\mathbb{R}^{d+1}$. In this case $V(\Delta) = \{e_1, \dots, e_{d+1}\}$ is the set of the vertices of Δ, where each e_i is the unit vector on the i-th axis, the barycentric coordinates with respect to $V(\Delta)$ coincide with the regular ones.

Assume $a = (a_1, \dots, a_{d+1})$, and $x = (x_1, \dots, x_{d+1})$. Since the point a lies in the interior of Δ, we have $a_i > 0$, for all $i = 1, \dots, d+1$, so we can consider fractions $x_1/a_1, \dots, x_{d+1}/a_{d+1}$.

Lemma 2.4. *In the situation above, we have $x \in \Delta_i(a)$ if and only if the minimum $\min_{1 \leqslant j \leqslant d+1} x_j/a_j$ is achieved at x_i/a_i. In particular, set $I := \{i \mid x_i/a_i = \min_{1 \leqslant j \leqslant d+1} x_j/a_j\}$, then I indexes the simplex of $\mathrm{sd}(\mathcal{F}(\Delta), a)$ whose interior contains x, i.e., using our notations, $\Delta_I(a) = \sigma(\mathrm{sd}(\mathcal{F}(\Delta), a), x)$.*

For example, taking $I=[d+1]$ we get $x_1/a_1=\cdots=x_{d+1}/a_{d+1}$; since $a_1+\cdots+a_{d+1}=x_1+\cdots+x_{d+1}=1$ we then get $x_i=a_i$ for all i, and so $x=a$.

Proof of Lemma 2.4. Given $1\leqslant i\leqslant d+1$, set $V:=\{a\}\cup V(\Delta)\setminus\{e_i\}$. The points of $\Delta_i(a)$ are given by all the convex linear combinations of V. Pick a specific point $x=(x_1,\dots,x_{d+1})\in\Delta_i(a)$, $x=\lambda_a a+\sum_{\substack{j=1,\dots,n\\ j\neq i}}\lambda_j e_j$. Then

$$\frac{x_j}{a_j}=\begin{cases}\lambda_a, & \text{for } i=j;\\ \lambda_a+\lambda_j/a_j, & \text{for } i\neq j.\end{cases}$$

Since $\lambda_j\geqslant 0$, for all j, we see that $\min_{1\leqslant j\leqslant d+1}x_j/a_j$ as achieved by x_i/a_i, as well as all x_j/a_j, for which $\lambda_j=0$. This proves all the statements of the lemma. $\square$

2.4. Points in general position and discriminants

Let again Δ denote the standard simplex in $\mathbb{R}^{d+1}$. Assume that $i,j\in[d+1]$, $i\neq j$, and $v\in\operatorname{int}\Delta$, $v=(v_1,\dots,v_{d+1})$. Let $\widetilde{H}(i,j,v)$ denote the hyperplane spanned by v and the points $V(\Delta)\setminus\{e_i,e_j\}$; that hyperplane is given by the equation $x_i/v_i=x_j/v_j$. Let H_e denote the hyperplane given by the equation $x_1+\cdots+x_{d+1}=1$, and set $H(i,j,v):=H_e\cap\widetilde{H}(i,j,v)$. Clearly, $H(i,j,v)$ is an affine hyperplane in H_e. Let $H^+(i,j,v)$ denote the closed halfspace in H_e bounded by $H(i,j,v)$ which does not contain the vertex e_i; it consists of all points $(x_1,\dots,x_{d+1})\in H_e$, such that $x_i/v_i\leqslant x_j/v_j$.

Given $1\leqslant i<j\leqslant d+1$, we set

$$D_{i,j}(v):=\operatorname{int}\Delta\cap H(i,j,v)\cap\bigcap_{\substack{1\leqslant k\leqslant d+1\\ k\neq i,j}}H^+(k,i,v). \tag{2.1}$$

In other words, $D_{i,j}(v)$ consists of all the points $x=(x_1,\dots,x_{d+1})$ of $\operatorname{int}\Delta$, satisfying

$$\frac{x_i}{v_i}=\frac{x_j}{v_j}\geqslant\frac{x_k}{v_k},\quad \text{for all } k\in[d+1]\setminus\{i,j\}. \tag{2.2}$$

From this description it is clear that $D_{i,j}(v)$ is a bounded polyhedron of dimension $d-1$. Let us now describe its polyhedral structure. For all $I\subseteq[d+1]$, such that $|I|\leqslant d$ we set

$$w_I(v):=\operatorname{span}\bigl(v,\{e_i\}_{i\in I}\bigr)\cap\operatorname{span}\bigl(\{e_i\}_{i\notin I}\bigr),$$

where the spans are taken inside of the hyperplane H_e. The coordinates of $w_I(v)$ are obtained by setting the coordinates of v indexed by the set I to 0, and scaling the rest of the coordinates so as to obtain the sum 1. In particular, we have $w_\emptyset(v):=v$, and $w_{[d+1]\setminus\{i\}}(v)=e_i$, for all $i\in[d+1]$.

With these notations, we see that the set of vertices of the polyhedron $D_{i,j}(v)$ is precisely $\{w_J(v)\}_J$, where we take all J satisfying $J\subseteq[d+1]\setminus\{i,j\}$. More generally, for an arbitrary $I\subseteq[d+1]$, such that $I\neq\emptyset$, we set

$$D_I(v):=\operatorname{conv}\bigl(\bigl\{w_J(v)\bigr\}_J\bigr), \tag{2.3}$$

where the index of $w_J(v)$ ranges over all subsets $J\subseteq[d+1]$, such that $J\cap I=\emptyset$. In particular, $D_{[d+1]}(v)=v$, while $D_{\{i\}}(v)$ is the star of e_i in $\operatorname{bd}(\Delta,f)$, with $f:\Delta\setminus\{\emptyset\}\to|\Delta|$ given by $f(I)=w_I(v)$. It is easy to see that $\dim D_I(v)=d+1-|I|$, specifically $D_I(v)$ consists of points $x=(x_1,\dots,x_{d+1})$ satisfying

$$\frac{x_i}{v_i}=\frac{x_j}{v_j},\quad \text{for all } i,j\in I,\quad \text{and}\quad \frac{x_i}{v_i}\geqslant\frac{x_k}{v_k},\quad \text{for all } i\in I,\ k\notin I. \tag{2.4}$$

Furthermore, it follows from (2.3) that

$$\bigcap_{j=1}^{k}D_{I_j}(v)=D_{\bigcup_{j=1}^{k}I_j}(v), \tag{2.5}$$

for arbitrary nonempty sets $I_1,\dots,I_k\subseteq[d+1]$, for all $j=1,\dots,k$. We shall need the following properties of the polyhedra $D_I(v)$.

Proposition 2.5. *Assume x and v are points in the interior of a d-simplex, and the subset $I\subseteq[d+1]$ is nonempty, then we have*

$$x\in D_I(v)\quad\Leftrightarrow\quad v\in\Delta_I(x). \tag{2.6}$$

Furthermore, whenever $x\in\Delta_I(v)$, we have $D_I(v)\subseteq D_I(x)$.

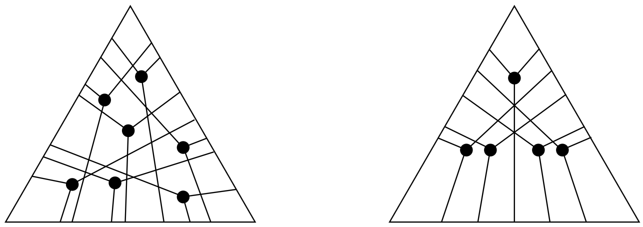

Fig. 2.1. Discriminants of point configurations: the left point configuration is in general position, whereas the right one is not.

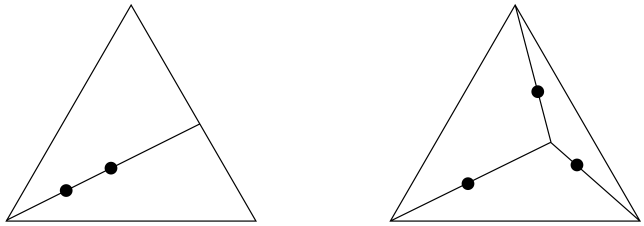

Fig. 2.2. The forbidden point configurations for $d = 2$.

Proof. The equivalence in (2.6) is obtained by comparing the statement of Lemma 2.4 with the coordinate description of $D_I(v)$ which we gave in (2.4).

Assume now $x \in \Delta_I(v)$, and let $x = (x_1, \dots, x_{d+1})$, $v = (v_1, \dots, v_{d+1})$. Pick $y \in D_I(v)$, $y = (y_1, \dots, y_{d+1})$. Since $x \in \Delta_I(v)$, we see that the minimum of $\{x_i/v_i\}_{i\in[d+1]}$ is achieved by $i \in I$. On the other hand, since $y \in D_I(v)$, we see that the maximum of $\{y_i/v_i\}_{i\in[d+1]}$ is achieved by $i \in I$. Combining these yields that the maximum of $\{y_i/x_i\}_{i\in[d+1]}$ is achieved by $i \in I$, implying that $y \in D_I(x)$. $\square$

Definition 2.6. Given a d-simplex Δ, and a collection S of points in the interior of Δ, we set

$$D(v) := \bigcup_{1\leqslant i<j\leqslant d+1} D_{i,j}(v) \quad \text{and} \quad D(S) := \bigcup_{v\in S} D(v).$$

We call $D(v)$ the **discriminant of the point** v, and we call $D(S)$ the **discriminant of the point configuration** S.

We see that $D(v)$ is isomorphic to a cubical complex, that all the maximal cubes in $D(v)$ contain the vertex v, and that the link of v in $D(v)$ is isomorphic to the $(d-2)$-dimensional skeleton of the d-simplex. Furthermore, we set $\widehat{D(S)} := (\operatorname{int}\Delta) \setminus D(S)$. See Fig. 2.1 for examples of discriminants of points and point configurations.

Definition 2.7. Given a d-simplex Δ, and a collection S of points in the interior of Δ, we say that points S are in **general position** if and only if the discriminants $D(v)$, for $v \in S$, intersect transversely. Specifically, for any $\widetilde{S} \subset S$, we require that

$$\dim \bigcap_{v\in\widetilde{S}} D(v) \leqslant d - |\widetilde{S}|. \tag{2.7}$$

If $d = 1$ any point configuration is in general position. If $d = 2$, the points of S are in general position if and only if two conditions are satisfied, see Fig. 2.2:

(1) any line spanned by a point from S and a vertex of Δ contains no other points from S;
(2) there does not exist a constellation of three points $x_1, x_2, x_3 \in S$, such that the three lines $\operatorname{span}(e_1, x_1)$, $\operatorname{span}(e_2, x_2)$, and $\operatorname{span}(e_3, x_3)$ intersect in one point, which does not belong to any of the convex hulls $\operatorname{conv}(e_1, x_1)$, $\operatorname{conv}(e_2, x_2)$, and $\operatorname{conv}(e_3, x_3)$.

Let $\mathcal{D}(n) \subset \Delta^n$ denote the locus of point configurations which are *not* in general position. We call $\mathcal{D}(n)$ the stellar discriminant. While the stellar discriminant is defined geometrically, and Eqs. (2.2) additionally provide a precise algebraic description, it would be useful to have a more combinatorial description, in the spirit of arrangement theory, see [9], of that algebraic variety, and possibly use that to compute associated algebro-topological invariants. Being an interesting and probably hard problem it lies outside the scope of the current investigation.

2.5. Bifurcations

First, we define the point distribution function $\xi_{v,S} : 2^{[d+1]} \setminus \{\emptyset\} \to 2^S$ by setting

$$\xi_{v,S}(I) := S \cap \operatorname{int} \Delta_I(v).$$

Obviously, the sets $\xi_{v,S}(I)$ are disjoint, and we have $\sum_{I \subseteq [d+1]} |\xi_{v,S}(I)| = n$. Since usually the set S will be fixed, we shall sometimes omit it as index and simply write ξ_v. Reversely, we define the point location function $\varphi_{v,S} : S \to 2^{[d+1]} \setminus \{\emptyset\}$, by setting $\varphi_{v,S}(x)$ to be the set $I \subseteq [d+1]$ such that $x \in \operatorname{int} \Delta_I(v)$. Clearly, we have $\xi_{v,S}(I) = \varphi_{v,S}^{-1}(I)$. Again, when S is clear, we write φ_v instead of $\varphi_{v,S}$.

Fix the set S, the vertex $v = (v_1, \dots, v_{d+1}) \in \operatorname{int} \Delta$, and let $\emptyset \neq A \subset [d+1]$. Let $\vec{w}^A = (w_1^A, \dots, w_{d+1}^A)$ be the vector in $\mathbb{R}^{d+1}$ defined by

$$w_i^A = \begin{cases} -v_i, & \text{if } i \in A, \\ c v_i, & \text{if } i \notin A, \end{cases} \tag{2.8}$$

for all $i \in [d+1]$, where the constant c is chosen so that $\vec{w}^A$ lies on the hyperplane $x_1 + \cdots + x_{d+1} = 0$, i.e., $c := \sum_{i \in A} v_i / \sum_{i \notin A} v_i$.

Proposition 2.8. *Given the set S, the vertex v, and the set $\emptyset \neq A \subset [d+1]$ as above, there exists a constant $N > 0$, such that for any $0 < \varepsilon < N$, setting $\tilde{v} := v + \varepsilon \vec{w}^A$ will yield the formulae*

$$\varphi_{\tilde{v}}(x) = \begin{cases} \varphi_v(x), & \text{if } \varphi_v(x) \subseteq A, \\ \varphi_v(x) \cap ([d+1] \setminus A), & \text{otherwise}, \end{cases} \tag{2.9}$$

for all $x \in S$, and hence accordingly

$$\xi_{\tilde{v}}(I) = \begin{cases} \xi_v(I), & \text{if } I \subseteq A, \\ \bigcup_{I \subseteq T \subseteq I \cup A} \xi_v(T), & \text{if } I \cap A = \emptyset, \\ 0, & \text{otherwise}, \end{cases} \tag{2.10}$$

for all $I \subseteq [d+1]$.

Proof. Eq. (2.10) follows directly from Eq. (2.9), so we just prove the latter. By the choice of $\vec{w}$, the point $\tilde{v}$ will lie in the hyperplane H_e for any choice of ε. To start with we assume that N is sufficiently small so that $\tilde{v}$ still lies in the interior of Δ.

Assume $v = (v_1, \dots, v_{d+1})$ and $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_{d+1})$. We have

$$\tilde{v}_i = \begin{cases} v_i(1-\varepsilon), & \text{if } i \in A; \\ v_i(1+c\varepsilon), & \text{if } i \notin A. \end{cases} \tag{2.11}$$

In particular,

$$\frac{v_i}{v_j} = \frac{\tilde{v}_i}{\tilde{v}_j}, \quad \text{if both } i, j \in A \text{ or both } i, j \notin A. \tag{2.12}$$

According to Lemma 2.4, $\max_{1 \leqslant i \leqslant d+1} v_i/x_i$ is achieved on the index set $\varphi_v(x)$. If $\varphi_v(x) \subseteq A$, then (2.11) together with (2.12) implies that $\tilde{v}_i/x_i = \tilde{v}_j/x_j$, whenever $i, j \in \varphi_v(x)$. We see that $\max_{1 \leqslant i \leqslant d+1} \tilde{v}_i/x_i$ is still achieved on the same index set $\varphi_v(x)$, as long as ε is chosen so small that $v_i/x_i > v_j/x_j$ implies $\tilde{v}_i/x_i > \tilde{v}_j/x_j$, for all $i, j \in [d+1]$.

We can thus write $\varphi_v(x) = B \cup C$, where $B = \varphi_v(x) \cap A$, $C = \varphi_v(x) \cap ([d+1] \setminus A)$, and assume that $C \neq \emptyset$. Again, choosing ε so small that $v_i/x_i > v_j/x_j$ implies $\tilde{v}_i/x_i > \tilde{v}_j/x_j$, for all $i, j \in [d+1]$, ensures that $\varphi_{\tilde{v}}(x) \subseteq \varphi_v(x)$. This time however, we get $\tilde{v}_i/x_i < \tilde{v}_j/x_j$, whenever $i \in B$, $j \in C$, and we get $\tilde{v}_i/x_i = \tilde{v}_j/x_j$, when $i, j \in C$. Hence we conclude that $\varphi_{\tilde{v}}(x) = C$, completing the verification of (2.9). □

For later use, we remark that, for fixed value of d, the complexity of finding a suitable constant N is $O(n^2)$.

Definition 2.9. When ε is taken to be as small as Proposition 2.8 requires, we shall call $\tilde{v}$ an A-**bifurcation** of the point v.

One can think of an A-bifurcation as "moving the point v a little bit away from vertices in A." The outcome of this will be that no vertices of S are "shared between A and its complement," i.e., for all $x \in S$, we have $\varphi_v(x) \subseteq A$ or $\varphi_v(x) \subseteq [d+1] \setminus A$. If this was already the case, the A-bifurcation would not change the function φ_v. In such a situation, we may want to keep moving the point v away from A until the first change in the function φ_v.

Definition 2.10. Assume v is a vertex such that for all $x \in S$ the set $\varphi_v(x)$ is either contained in A or in its complement, and, additionally, there exists $x \in S$, such that $\varphi_v(x) \subseteq A$. Let $\vec{w}^A \in \mathbb{R}^{d+1}$ be given by (2.8), and let ε be the smallest positive number such that $\varphi_{\tilde{v}} \neq \varphi_v$ as a function, where $\tilde{v} = v + \varepsilon \vec{w}^A$. We call $\tilde{v}$ the A-**maxibifurcation** of v.

Unlike the case of a bifurcation, we cannot say precisely what happens to the function φ when a maxibifurcation is performed. The next proposition summarizes what can be said.

Proposition 2.11. *Assume $\tilde{v}$ is the A-maxibifurcation of v. Then $\varphi_{\tilde{v}}(x) = \varphi_v(x)$, for all $x \in S$ such that $\varphi_v(x)$ is contained in the complement of A, and $\varphi_{\tilde{v}}(x) \cap A = \varphi_v(x)$, for all $x \in S$ such that $\varphi_v(x)$ is contained in A. Furthermore, there exists $x \in S$ such that $\varphi_v(x) \subseteq A$ and $\varphi_{\tilde{v}}(x) \neq \varphi_v(x)$.*

Proof. The argument is similar to that of Proposition 2.8. The main difference is that ε is not small anymore, so it should happen that for some point $x \in S$, and for some $i, j \in [d+1]$ we have $v_i/x_i > v_j/x_j$, but $\tilde{v}_i/x_i = \tilde{v}_j/x_j$. Due to (2.11) this will not happen for a point $x \in S$, such that $\varphi_v(x) \subseteq [d+1] \setminus A$. Hence it will have to happen for one or several points $x \in S$, such that $\varphi_v(x) \subseteq A$. For these points the sets $\varphi_v(x)$ will increase by some indices from the complement of A. This proves all the statements of the proposition. $\square$

Clearly, the complexity of performing a maxibifurcation is $O(n)$ as well.

3. Stellar equipartitions of points in general position

3.1. Hypergraph associated to a stellar subdivision

Let $G_{v,S}$ denote the characteristic hypergraph of the function $\xi_{v,S}$. Specifically, we have $V(G_{v,S}) := [d+1]$, and the set of hyperedges is given by the following rule: for $T \subseteq [d+1]$, such that $|T| \geqslant 2$, we have $T \in E(G_{v,S})$ if and only if $|\xi_{v,S}(T)| \geqslant 1$. Clearly, $G_{v,S}$ has no hyperedges if and only if $v \in \widehat{D(S)}$.

Definition 3.1. We call a hypergraph G **acyclic** if there do not exist a sequence of vertices $i_1, \dots, i_m \in V(G)$, for $m \geqslant 2$, and a sequence of distinct hyperedges $T_1, \dots, T_m \in E(G)$, such that $i_1, i_m \in T_m$, and $i_j, i_{j+1} \in T_j$, for $j \in [m-1]$.

In particular, taking $m = 2$ in Definition 3.1, we see that the hyperedges of an acyclic hypergraph cannot have an intersection of cardinality more than 1, in particular, they cannot contain each other.

Lemma 3.2. *Let Δ be a geometric d-simplex, $d \geqslant 1$, and let S be collection of n points in general position in* $\operatorname{int}\Delta$. *Then the hypergraph $G_{v,S}$ is acyclic, and, furthermore, $|\xi_{v,S}(T)| \leqslant 1$ for all $T \subseteq [d+1]$ such that $|T| \geqslant 2$.*

Proof. Assume that we are given $i_1, \dots, i_m \in [d+1]$, $T_1, \dots, T_m \subseteq [d+1]$, and $x_1, \dots, x_m \in S$, such that the following conditions are true

- $x_i \neq x_j$, for all $i \neq j$, $i, j \in [d+1]$;
- $x_m \in \Delta_{i_1,i_m}(v)$, and $x_k \in \Delta_{i_k,i_{k+1}}(v)$, for all $1 \leqslant k \leqslant m-1$;
- $i_1, i_m \in T_m$, and $i_k, i_{k+1} \in T_k$, for all $1 \leqslant k \leqslant m-1$.

We want to show that this is impossible when the points of S are in general position. Since $x_k \in \Delta_{i_k,i_{k+1}}(v)$, for $k = 1, \dots, m-1$, and $x_m \in \Delta_{i_m,i_1}(v)$, by Proposition 2.5 we have $D_{i_1,i_2}(x_1) \supseteq D_{i_1,i_2}(v)$, $D_{i_2,i_3}(x_2) \supseteq D_{i_2,i_3}(v)$, $\dots$, $D_{i_{m-1},i_m}(x_{m-1}) \supseteq D_{i_{m-1},i_m}(v)$, and $D_{i_m,i_1}(x_m) \supseteq D_{i_m,i_1}(v)$. Therefore,

$$\left(\bigcap_{j=1}^{m-1} D_{i_j,i_{j+1}}(x_j)\right) \cap D_{i_m,i_1}(x_m) \supseteq \left(\bigcap_{j=1}^{m-1} D_{i_j,i_{j+1}}(v)\right) \cap D_{i_m,i_1}(v) = D_I(v). \tag{3.1}$$

Since the points of S are assumed to be in general position, the inequality (2.7) says that the dimension of the left hand side of (3.1) is less than or equal to $d-m$. On the other hand, $\dim D_I(v) = d+1-|I| = d+1-m$, yielding a contradiction.

Assume now that there exists a set $T \subseteq [d+1]$, such that $|T| \geqslant 2$, and $|\xi_{v,S}(T)| \geqslant 2$. We can then choose distinct $i_1, i_2 \in T$ and distinct $x_1, x_2 \in S$, such that $x_1, x_2 \in \Delta_{i_1,i_2}(v)$. Using the argument above for $m = 2$ we see that this is impossible.

Finally, assume that the hypergraph $G_{v,S}$ is not acyclic, and pick a sequence of vertices $i_1, \dots, i_m \in V(G)$, and a sequence of distinct hyperedges $T_1, \dots, T_m \in E(G)$, which satisfy conditions of Definition 3.1. Pick now $x_1, \dots, x_m \in S$ such that $\varphi_{v,S}(x_m) = T_m$, and $\varphi_{v,S}(x_k) = T_k$, for all $1 \leqslant k \leqslant m-1$. All the x_i's must be distinct, since they belong to disjoint sets $\operatorname{int}\Delta_{T_i}(v)$, for $i = 1, \dots, m$. Again this yields a contradiction with the argument above. $\square$

Definition 3.3. Given a collection of n points S in $\operatorname{int}\Delta$ in general position, and a point $v \in \operatorname{int}\Delta$, we say that v has **simple singularities** if the hypergraph $G_{v,S}$ does not have hyperedges of cardinality 3 and higher; in other words, it is simply a graph.

Before proceeding with the main theorem we need one more technical result.

Proposition 3.4. *Assume S is a collection of points in $\operatorname{int}\Delta$ in general position, and a point $v \in \operatorname{int}\Delta$ has simple singularities. Assume furthermore that $e = (i, j)$ is a bridge of $G_{v,S}$. Then, there exists a point $\tilde{v} \in \operatorname{int}\Delta$ which has simple singularities, such that $G_{\tilde{v},S}$ is a graph obtained from $G_{v,S}$ by deleting the edge e. Moreover, let $x \in S$ be such that $\varphi_{v,S}(x) = \{i, j\}$, then $\varphi_{\tilde{v},S}(y) = \varphi_{v,S}(y)$ for all $y \in S$, $y \neq x$, and $\varphi_{\tilde{v},S}(x) = \{i\}$.*

Proof. Let A denote the set of vertices of $G_{v,S}$ which can be reached from the vertex i without using the edge e. We set $\tilde{v}$ to be an A-bifurcation of v. If $T \subseteq [d+1]$ such that $T \cap A \neq \emptyset$, $T \cap ([d+1] \setminus A) \neq \emptyset$, and $T \neq \{i, j\}$, then since e is a bridge we have $\xi_{v,S}(T) = 0$. The statement follows now directly from Proposition 2.8. $\square$

When performing the A-bifurcation described in the proof of Proposition 3.4 we shall say that we are *discharging the edge e towards the vertex i*.

3.2. Constructing equipartitions and near-equipartitions

For an arbitrary point $v \in \operatorname{int}\Delta$ we set $p_i(v, S) := |\xi_{v,S}(\{i\})|$, for all $1 \leqslant i \leqslant d+1$, and furthermore we set $E(v, S) := (p_1(v, S), \dots, p_{d+1}(v, S)) \in \mathbb{Z}^{d+1}_{\geqslant 0}$. For an arbitrary $E \in \mathbb{Z}^{d+1}_{\geqslant 0}$, $E = (p_1, \dots, p_{d+1})$, we set $\max(E) := \max_{1 \leqslant i \leqslant d+1} p_i$, and set

$$\operatorname{Ind}(E) := \bigl\{ i \in [d+1] \bigm| p_i(E) = \max(E) \bigr\},$$
$$\operatorname{Ind}^-(E) := \bigl\{ i \in [d+1] \bigm| p_i(E) = \max(E) - 1 \bigr\}.$$

Clearly, the set $\operatorname{Ind}(E)$ is always nonempty, while the set $\operatorname{Ind}^-(E)$ might be empty. Finally, let $\Omega(S)$ denote the set of all vectors $E(v, S)$, when v is any point in $\operatorname{int}\Delta$. We can now formulate the main theorem for the configurations of points in general position.

Theorem 3.5. *Let Δ be a geometric d-simplex, $d \geqslant 1$, and let S be collection of n points in $\operatorname{int}\Delta$ in general position. There exists a point a, such that the interior of each of the d-simplices of $\operatorname{sd}(\mathcal{F}(\Delta), a)$ contains precisely $\lfloor \frac{n}{d+1} \rfloor$ points from S.*

Proof. The case $d = 1$ is obvious, so assume that $d \geqslant 2$. Note that $\widehat{D(S)}$ is a disjoint union of open polyhedra, and the vector $E(a)$ depends only on the choice of such an open polyhedron, in which the point a is lying.

Let $\Gamma \in \operatorname{int}\Delta$ be the set consisting of all points v, which have simple singularities, and such that whenever (i, j) is an edge of $G_{v,S}$, then either $i, j \in \operatorname{Ind}(E(v, S))$, or $i, j \in \operatorname{Ind}^-(E(v, S))$; i.e.,

$$E(G_{v,S}) \subseteq \operatorname{Ind}\bigl(E(v, S)\bigr) \times \operatorname{Ind}\bigl(E(v, S)\bigr) \cup \operatorname{Ind}^-\bigl(E(v, S)\bigr) \times \operatorname{Ind}^-\bigl(E(v, S)\bigr).$$

Clearly, $\Gamma \supseteq \widehat{D(S)}$, in particular, Γ is not an empty set. For $v \in \Gamma$, let $G^m_{v,S}$ denote the subgraph of $G_{v,S}$ induced by the vertices $\operatorname{Ind}(E(v, S))$, and let $G^-_{v,S}$ denote the subgraph of $G_{v,S}$ induced by the vertices $\operatorname{Ind}^-(E(v, S))$. Since the points of S are in general position, Lemma 3.2 implies that both graphs $G^m_{v,S}$ and $G^-_{v,S}$ are forests.

We now pick $v \in \Gamma$, such that

(1) $\max(E(v, S))$ is the minimal possible among all choices $v \in \Gamma$;
(2) the cardinality $|E(G^m_{v,S})|$ is minimal possible among those $v \in \Gamma$ which satisfy (1);
(3) the cardinality $|\operatorname{Ind}(E(v, S))|$ is minimal possible among those $v \in \Gamma$ which satisfy (1) and (2).

Clearly, such a choice of $v \in \Gamma$ can always be made, although it will usually not be unique.

Let v_1 be the $\operatorname{Ind}(E(v, S))$-maxibifurcation of v. The effect of a maxibifurcation has been described in Proposition 2.11. According to that, the only points of S where $\varphi_{v,S}$ may differ from $\varphi_{v_1,S}$ either lie in $\Delta_i(v)$ for $i \in \operatorname{Ind}(E(v, S))$ or in $\Delta_{i,j}(v)$, where (i, j) is an edge of $G^m_{v,S}$, and by Lemma 3.2, for every such a pair (i, j) there is at most 1 such vertex from S. Given the vertex v_1, we can bifurcate back some of these changes. Note that the graph $G_{v,S}$ does not change when we perform a C-bifurcation, where C is any connected component of $G_{v,S}$.

This means that we can perform a series of bifurcations to obtain v_2, such that there exist a connected component A of $G^m_{v,S}$ and a connected component B of $G_{v,S}$ outside of $G^m_{v,S}$, such that

- all the points $x \in S$ on which $\varphi_{v_2,S}$ differs from $\varphi_{v,S}$ lie either in $\Delta_i(v)$ for $i \in A$ or in $\Delta_{i,j}(v)$, where (i, j) is an edge of A;
- for these points, the difference $\varphi_{v_2,S}(x) \setminus \varphi_{v,S}(x)$ lies in B.

More specifically, let $x \in S$ be such that $\varphi_{v_1,S}(x) \neq \varphi_{v,S}(x)$. Let A be the connected component of $G^m_{v,S}$ which contains $\varphi_{v,S}(x)$ (the latter must be a vertex or an edge). Due to acyclicity of $G_{v_1,S}$, we have $\varphi_{v_1,S}(x) = \varphi_{v,S}(x) \cup \{l\}$, for some $l \in [d+1] \setminus \mathrm{Ind}(E(v,S))$. Let B be the connected component of $G_{v,S}$ containing l. Set $C := ([d+1] \setminus \mathrm{Ind}(E(v,S))) \cup A$ and $D := [d+1] \setminus (\mathrm{Ind}(E(v,S)) \cup B)$. The vertex v_2 can now be constructed from v_1 by first taking a C-bifurcation and then taking a D-bifurcation of the obtained vertex.

Even stronger, since the hypergraph $G_{v_2,S}$ is acyclic, we must have one of the two situations:

Case I. There exist $k \in \mathrm{Ind}(E(v,S))$, $l \in [d+1] \setminus \mathrm{Ind}(E(v,S))$, and $x \in S$, such that $\varphi_{v,S}(x) = \{k\}$, $\varphi_{v_2,S}(x) = \{k,l\}$, and $\varphi_{v,S}(y) = \varphi_{v_2,S}(y)$, for all $y \in S \setminus \{x\}$.

Case II. There exist $k_1, k_2 \in \mathrm{Ind}(E(v,S))$, $l \in [d+1] \setminus \mathrm{Ind}(E(v,S))$, and $x \in S$, such that $\varphi_{v,S}(x) = \{k_1,k_2\}$, $\varphi_{v_2,S}(x) = \{k_1,k_2,l\}$, and $\varphi_{v,S}(y) = \varphi_{v_2,S}(y)$, for all $y \in S \setminus \{x\}$.

We start by settling Case I. In this situation we have $E(v_2,S) = E(v,S) - e_k$, and $G_{v_2,S}$ is obtained from $G_{v,S}$ by adding the edge (k,l). We break up our argument in considering 4 cases.

Case 1. Assume that $l \notin \mathrm{Ind}^-(E(v,S))$ and k is an isolated vertex of $G^m_{v,S}$.

Let v_3 be obtained from v_2 by discharging the edge (k,l) to l. Then p_l goes up by 1, but since $l \notin \mathrm{Ind}^-(E(v,S))$ that does not change $\mathrm{Ind}(E(v,S))$. On the other hand, since k had no edges connecting it to other vertices, the vertex v_3 lies in Γ. If $|\mathrm{Ind}(E(v,S))| = 1$, then $\max(E(v,S)) > \max(E(v_3,S))$. If $|\mathrm{Ind}(E(v,S))| \geqslant 2$, then $\max(E(v,S)) = \max(E(v_3,S))$, $|E(G^m_{v,S})| = |E(G^m_{v_3,S})|$, and $|\mathrm{Ind}(E(v,S))| > |\mathrm{Ind}(E(v_3,S))|$. In either case we get a contradiction with the minimality of the choice of v.

Case 2. Assume that $l \notin \mathrm{Ind}^-(E(v,S))$ and k is not an isolated vertex of $G^m_{v,S}$.

We choose v_3 as in the previous case. The difference is that now v_3 does not lie in Γ, as there are edges connecting k, which is in $\mathrm{Ind}^-(E(v_3,S))$, to vertices in $\mathrm{Ind}(E(v_3,S))$. We rectify the situation by choosing some $\tilde{k} \in \mathrm{Ind}(E(v_3,S))$, such that $(k,\tilde{k}) \in G_{v_3,S}$, and discharging the edge $(k,\tilde{k})$ to k. The obtained vertex v_4 is in Γ, since $k \in \mathrm{Ind}(E(v_4,S))$. We have $\max(E(v,S)) = \max(E(v_4,S))$ and $|E(G^m_{v_4,S})| = |E(G^m_{v,S})| - 1$, yielding a contradiction to the choice of v.

Case 3. Assume that $l \in \mathrm{Ind}^-(E(v,S))$ and k is an isolated vertex of $G^m_{v,S}$.

In this case, already the vertex v_2 is in Γ. If $|\mathrm{Ind}(E(v,S))| = 1$, then $\max(E(v,S)) > \max(E(v_2,S))$. If $|\mathrm{Ind}(E(v,S))| \geqslant 2$, then $\max(E(v,S)) = \max(E(v_2,S))$, $|E(G^m_{v,S})| = |E(G^m_{v_2,S})|$, and $|\mathrm{Ind}(E(v,S))| > |\mathrm{Ind}(E(v_2,S))|$. In either case we get a contradiction with the minimality of the choice of v.

Case 4. Assume that $l \in \mathrm{Ind}^-(E(v,S))$ and k is not an isolated vertex of $G^m_{v,S}$.

This is the most interesting of the four cases. Again, the vertex v_2 is not in Γ. We have $k \in \mathrm{Ind}^-(E(v_2,S))$. Choose some $\tilde{k} \in \mathrm{Ind}(E(v_2,S))$, such that $(k,\tilde{k}) \in E(G_{v_2,S})$. Let T denote the connected component of $G_{v,S}$ containing l. We know that T is a tree. Take l to be the root of T, and orient all the edges in T away from l. Let v_3 be obtained from v_2 by the following sequence of discharges: discharge the edge $(k,\tilde{k})$ to k, then discharge the edge (k,l) to l, then, starting from l and following standard breadth-first search, we discharge all the edges of T to their respective target vertices. Note that $|\mathrm{Ind}(E(v_3,S))| = |\mathrm{Ind}(E(v,S))| + |T|$. We have $\max(E(v,S)) = \max(E(v_3,S))$, and $|E(G^m_{v_3,S})| = |E(G^m_{v,S})| - 1$, yielding a contradiction to the choice of v.

Let us now settle Case II. Here we have $(k_1,k_2) \in E(G^m_{v,S})$, and $G_{v_2,S}$ is obtained from $G_{v,S}$ by replacing (k_1,k_2) with the hyperedge $\{k_1,k_2,l\}$. Let v_3 be the $\mathrm{Ind}(E(v,S))$-bifurcation of v_2. The point x will move from $\Delta_{k_1,k_2,l}(v_2)$ to $\Delta_{\{l\}}(v_3)$, so v_3 has only simple singularities. We can describe the function $\xi_{v_3,S}$ directly:

- $\xi_{v_3,S}(T) = \xi_{v,S}(T)$, for all $T \neq \{k_1,k_2\}, \{l\}$;
- $\xi_{v_3,S}(\{k_1,k_2\}) = \emptyset$;
- $\xi_{v_3,S}(\{l\}) = \xi_{v,S}(\{k_1,k_2\}) = \{x\}$.

If $l \in \mathrm{Ind}^-(E(v,S))$, then v_3 may not lie in Γ. We fix that in the same way as in the case 4 above. Namely, assume C is the connected component of $G_{v,S}$ containing the vertex l. Since $G_{v,S}$ is acyclic, C must be a tree. Choose l to be the root of that tree, and orient all the edges of C away from the root. Let v_4 be the result of discharging all edges of C in their respective directions. If C consists of l only, or if $l \notin \mathrm{Ind}^-(E(v_2,S))$, we just set $v_4 := v_3$. We have $v_4 \in \Gamma$. We can see now that $\max(E(v_4,S)) = \max(E(v,S))$, but $|E(G^m_{v_4,S})| = |E(G^m_{v,S})| - 1$, leading to a contradiction with the choice of v. $\square$

When d is fixed, the algorithm in the proof of Theorem 3.5 will reduce $\max(E(v,S))$ in a constant number of steps. Therefore, the total number of steps is $O(n)$, and we see that the complexity of the algorithm is $O(n^2)$.

Corollary 3.6. *Let Δ be a geometric d-simplex, $d \geqslant 1$, and let S be collection of n points in $\mathrm{int}\,\Delta$ in general position. For an arbitrary $\pi = (\pi_1, \ldots, \pi_{d+1}) \in \mathbb{Z}^{d+1}_{\geqslant 0}$, such that $\pi_1 + \cdots + \pi_{d+1} = n$, there exists a point a_π, such that $p(a_\pi, S) = \pi$; i.e.,*

$$\Omega(S) \supseteq \left\{(x_1, \ldots, x_{d+1}) \in \mathbb{Z}^{d+1}_{\geqslant 0} \;\middle|\; x_1 + \cdots + x_{d+1} = |S|\right\}.$$

Proof. Apply the argument of Theorem 3.5, replacing the function $E : \mathrm{int}\,\Delta \to \mathbb{Z}^{d+1}_{\geqslant 0}$ by the function $E - \pi$. $\square$

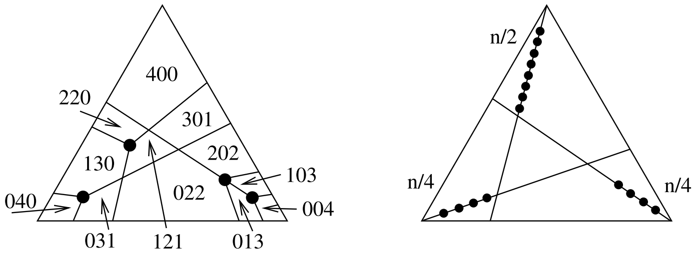

Fig. 4.1. The smallest example with no stellar equipartition, and the example with a linear gap in the stellar distribution.

Corollary 3.7. *Let Δ be a geometric d-simplex, $d \geqslant 1$, and let S be collection of n points in general position from Δ. Then the number of connected components of $\widehat{D(S)}$ is equal to $\binom{n+d}{d}$.*

Proof. Given $v_1, v_2 \in \widehat{D(S)}$, such that $E(v_1, S) = E(v_2, S)$, it follows from Lemma 2.4 that also for any convex combination $w = \lambda v_1 + (1-\lambda)v_2$, $1 \geqslant \lambda \geqslant 0$, we have $E(w, S) = E(v_1, S)$. This means that different connected components of $\widehat{D(S)}$ must have different vectors $E(-, S)$. Since, by Corollary 3.6, any such vector is realized. We conclude that it is realized by precisely one component. From elementary enumerative combinatorics we know that the number of vectors $\pi = (\pi_1, \dots, \pi_{d+1}) \in \mathbb{Z}^{d+1}_{\geqslant 0}$, such that $\pi_1 + \cdots + \pi_{d+1} = n$, is $\binom{n+d}{d}$, hence so is the number of connected components of $\widehat{D(S)}$. $\square$

4. Points in non-general position

4.1. Point sets with linear gap in the stellar distribution

When $d = 1$ any configuration of points is in general position. For $d = 2$, any configuration of at most 3 points can be equipartitioned by a stellar subdivision. On the left hand side of Fig. 4.1 we give the smallest example of a configuration of points which are not in general position, for which there is no stellar equipartition. On the right hand side of Fig. 4.1 we give an example of a configuration of points with a linear gap in the stellar distribution.

4.2. Stellar properly reducing partitions

As mentioned above, if the points are not in general position then no equipartition needs to exist. Even more, we cannot guarantee the existence of a stellar partition $\mathrm{sd}(\mathcal{F}(\Delta), v)$ where the differences between $p_i(v, S)$ do not exceed a fixed constant.

On the positive side, we can always find a stellar subdivision avoiding many points, falling within the same simplex.

Definition 4.1. Let Δ be a d-simplex, S is a collection of points in $\operatorname{int}\Delta$, and $v \in \operatorname{int}\Delta$. We say that the stellar subdivision $\mathrm{sd}(\mathcal{F}(\Delta), v)$ is **properly reducing** if $\max(E(v, S)) \leqslant n/(d+1)$.

We now set

$$\operatorname{mmax}(S) := \min_{v \in \Delta} \max\bigl(E(v, S)\bigr). \tag{4.1}$$

Let Σ denote the set of points $v \in \Delta$, such that $\max(E(v, S)) = \operatorname{mmax}(S)$. If the point $\tilde{v}$ is sufficiently close to the point v, then $E(\tilde{v}, S) \geqslant E(v, S)$. Hence, the complement of Σ is open, and therefore Σ is compact.

Consider further the function $\eta : \Delta \to (0, \infty)$, where $\eta(v)$ is equal to the volume of $\bigcup_{i \in \operatorname{Ind}(E(v,S))} \Delta_i$. The function η is left-continuous in the following sense: for any point $v \in \Delta$, and any positive number ε there exists an open neighborhood $N(v)$ of v, such that $\eta(N(v)) \subseteq (\eta(v) - \varepsilon, \infty)$. This is because changing v sufficiently little will only possibly increase values in $E(v, S)$, hence $\operatorname{Ind}(E(\tilde{v}, S)) \supseteq \operatorname{Ind}(E(v, S))$, for all $\tilde{v} \in N(v)$, and the volume of each Δ_i, $i \in \operatorname{Ind}(E(v, S))$, changes little.

Theorem 4.2. *For a given d-simplex Δ, and an arbitrary collection of points S in* $\operatorname{int}\Delta$, *we have* $\operatorname{mmax}(S) \leqslant \lfloor \frac{n}{d+1} \rfloor$, *in other words, there always exists a properly reducing stellar subdivision.*

Proof. Since Σ is compact and η is left-continuous, the restriction $\eta|_\Sigma$ achieves its minimum at some point in Σ. Assume that this minimum is achieved at $v \in \operatorname{span}(V(\Delta) \setminus \{e_i\})$, for some $1 \leqslant i \leqslant d+1$. Let $\tilde{v}$ be the $([d+1] \setminus \{i\})$-bifurcation of v. Since Δ_i is degenerate, it does not contain any points from S. Therefore, $E(\tilde{v}, S) = E(v, S)$, and $\eta(\tilde{v}) < \eta(v)$, yielding a contradiction to the choice of v.

Let us now take $v \in \Sigma \cap \operatorname{int}\Delta$, such that $\max(E(v,S)) = \operatorname{mmax}(S)$, and $\eta(v) \leqslant \eta(w)$, for all $w \in \Sigma$. We phrase our argument as an algorithmic procedure. We start by setting $v_0 := v$, $I_1 := \operatorname{Ind}(E(v,S))$, and $k := 1$. As long as $I_1 \cup \dots \cup I_k \neq [d+1]$ we repeat the following step. Set $A_k := I_1 \cup \dots \cup I_k$, and let v_k be an A_k-bifurcation of v. Furthermore, set

$$I_{k+1} := \{ i \mid p_i(v_k, S) \geqslant \operatorname{mmax}(S) \} \setminus A_k.$$

If I_{k+1} is empty, then $\operatorname{Ind}(E(v_k,S)) = \operatorname{Ind}(E(v,S))$, as $p_i(v_k,S) = p_i(v,S)$ for all $i \in A_k$, implying that $\eta(v_k) < \eta(v)$. Since also $\max(E(v_k,S)) = \max(E(v,S)) = \operatorname{mmax}(S)$ we obtain a contradiction to the minimality of $\eta(v)$. Thus, we may assume that the set I_{k+1} is nonempty.

From the general properties of bifurcations we have

$$\sum_{T \in J_i} |\xi_v(T)| = p_i(v_k, S) \geqslant \operatorname{mmax}(S), \quad \text{for all } i \in I_{k+1}, \tag{4.2}$$

where J_i denotes the closed interval $[\{i\}, A_k \cup \{i\}]$ in the Boolean lattice $\mathcal{B}_{d+1}$. We now increase k by 1 and repeat the above step. Since at each step the set I_{k+1} is nonempty, the process will terminate after finitely many steps.

Assume now the process has terminated after s steps, yielding the partition $I_1 \cup \dots \cup I_s = [d+1]$, and hence also intervals $J_1, \dots, J_{d+1}$. Clearly, these intervals are disjoint, hence using (4.2) we obtain

$$n = \sum_{T \subseteq [d+1]} |\xi_v(T)| \geqslant \sum_{i=1}^{d+1} \sum_{T \in J_i} |\xi_v(T)| \geqslant \sum_{i=1}^{d+1} \operatorname{mmax}(S) = (d+1) \operatorname{mmax} S,$$

implying $\frac{n}{d+1} \geqslant S$. □

We remark that the bound $n/(d+1)$ in Theorem 4.2 is optimal, as the following example demonstrates. Let k be any positive integer, and let S consist of $k(d+1)$ points in general position. Since each point of $\operatorname{int}\Delta$ lies in the intersection of at most d discriminants of points from S, at most d points from S will lie on the $(d-1)$-skeleton of any stellar subdivision. Thus, there will be at least $kd+1$ points from S left to be distributed among the $d+1$ d-simplices formed by the stellar subdivision, implying that at least one of the simplices will have to contain at least k points from S.

When actually looking for a point v such that $\max(E(v,S)) = \operatorname{mmax}(S)$ one can always restrict oneself to the following set of finitely many points.

Definition 4.3. Let Δ be a d-simplex, S is a collection of points in $\operatorname{int}\Delta$. We call 0-skeleton of $D(S)$ the **secondary configuration** associated to S.

As we noticed above, when the point v is moving along the same open polyhedra in the polyhedral structure of $D(S)$, the vector $E(v,S)$ remains constant; furthermore when v moves to a lower-dimensional polyhedron on the boundary, some of the entries of $E(v)$ may get smaller, as some points of S may move to the d-skeleton of $\operatorname{sd}(\mathcal{F}(\Delta), v)$, other entries of $E(v,S)$ will remain the same. In particular, this implies that there exists a point v in the secondary configuration associated to S, such that $\operatorname{mmax}(S) = \max(E(v,S))$.

5. Applications of stellar equipartitions

5.1. *Equipartitions by a tropical hyperplane*

We shall now apply our results in the context of tropical geometry, see [6,10]. Recall that a *tropical semiring* is the triple $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, where the operations $\oplus$ and $\odot$ are defined by

$$x \oplus y := \min(x, y), \qquad x \odot y := x + y.$$

A *tropical monomial* is a function $a \odot x_1^{\odot \alpha_1} \odot \dots \odot x_d^{\odot \alpha_d} = a + \alpha_1 x_1 + \dots + \alpha_d x_d$. A *tropical polynomial* is a finite sum of tropical monomials, that is a function

$$\begin{aligned} p(x_1, \dots, x_d) &= a \odot x_1^{\odot \alpha_1} \odot \dots \odot x_d^{\odot \alpha_d} + b \odot x_1^{\odot \beta_1} \odot \dots \odot x_d^{\odot \beta_d} + \cdots \\ &= \min(a + \alpha_1 x_1 + \dots + \alpha_d x_d, b + \beta_1 x_1 + \dots + \beta_d x_d, \dots). \end{aligned} \tag{5.1}$$

To each tropical polynomial one can associate the set of *tropical zeros* $Z(p)$: it consists of those points of $\mathbb{R}^d$, where the minimum in (5.1) is attained at least once; this is precisely the locus of the points where the function fails to be linear.

The *tropical line* is a set of tropical zeros of a tropical linear function in two variables, that is $Z(c \oplus a \odot x \oplus b \odot y)$. An example of a tropical line is shown on the left hand side of Fig. 5.1. It is easy to see that all the tropical lines are obtained from this example by parallel translations. More generally, a *tropical hyperplane* is a set of tropical zeros of a tropical linear

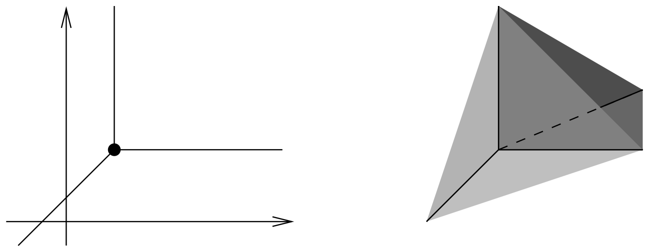

Fig. 5.1. A tropical line and a tropical hyperplane.

function in d variables, that is $Z(c \oplus a_1 \odot x_1 \oplus \cdots \oplus a_d \odot x_d)$. An example of a tropical hyperplane is shown on the right hand side of Fig. 5.1. Again, all tropical hyperplanes are parallel translates of this example.

Given a point configuration S in $\mathbb{R}^d$ and a tropical hyperplane H, let $p(H,S)$ denote the $(d+1)$-tuple recording the distribution of points from S in the open cones $\mathbb{R}^d \setminus H$. Of course, here we need to choose an order on the connected components of $\mathbb{R}^d \setminus H$; for our purposes any fixed (independent on H) order will do.

Theorem 5.1. *Assume S is a collection of n points in $\mathbb{R}^d$.*

(1) *If the points of S are in general position, then there exists a tropical hyperplane H, such that $p(H,S) = (\lfloor \frac{n}{d+1} \rfloor, \dots, \lfloor \frac{n}{d+1} \rfloor)$.*
(2) *Assume again the points of S are in general position. For an arbitrary ordered partition $\pi = (\pi_1, \dots, \pi_{d+1}) \in \mathbb{Z}^{d+1}_{\geqslant 0}$, such that $\pi_1 + \cdots + \pi_{d+1} = n$, there exists a tropical hyperplane H_π, such that $p(H_\pi, S) = \pi$.*

Proof. For an arbitrary positive number N, let Δ_N denote the simplex whose vertices have coordinates $(N,0,\dots,0)$, $(0,N,0,\dots,0), \dots, (0,\dots,0,N)$, and $(-N,\dots,-N)$. Assume first the points in S are in general position. Obviously, for sufficiently large N the points from S will lie in the interior of Δ_N. Furthermore, the combinatorial type of the stratification of Δ_N by the discriminants of points from S, stabilizes for large N as well. The statements (1) and (2) follow immediately from the analogous simplicial statements of Theorem 3.5 and Corollary 3.6. □

5.2. Fast barycentric embedding of a 0-dimensional subcomplex

Let us now return to the question of fast barycentric embeddings which was considered in the Introduction. We restrict ourselves to the case $\dim L = 0$. The case $\dim L \geqslant 1$ is more complicated, see [8] for some conjectures.

When $\dim L = 0$, L is just a finite set of points. Assume that $|L| = n$, that $K = \Delta^d$ is a d-simplex, and that all the points from L lie in the interior of K. If the points of L are in general position, then, by Corollary 3.6 there exists a point $a \in \operatorname{int} K$, such that all the points of L are contained in the interiors of the d-simplices $\Delta_1, \dots, \Delta_{d+1}$ of $\operatorname{sd}(K,a)$, and each of these open d-simplices contains at most $\lceil \frac{n}{d+1} \rceil$ points from L. Furthermore, since the locus of all such points a is an open set, we can pick a so that the points $L \cap \Delta_i$ are in general position inside Δ_i, for every $i = 1, \dots, d+1$.

Proposition 5.2. *Assume $\dim L = 0$, $K = \Delta^d$, $|L| \subset \operatorname{int} K$, and points of L are in general position in K, then there exists an iterated barycentric subdivision $\operatorname{bd}^t K$, where $t = \lceil \log_{(d+1)!} n \rceil + 1$, such that L is a subcomplex of $\operatorname{bd}^t K$.*

Note that for $d = 1$, we have the precise bound of $t = \lfloor \log_2 n \rfloor + 1$.

Proof of Proposition 5.2. Iterate the subdivision procedure above using the standard presentation of the barycentric subdivision as a sequence of stellar subdivisions, which we mentioned in Section 2.2. □

Assume now that the points from L are not in general position. Then it is possible to perform two barycentric subdivisions, after which the points are in general position. Indeed, for the first subdivision pick the points in the small neighborhoods of the actual barycenters. The new faces can be labeled by sequences $F_0 \subset F_1 \subset \cdots \subset F_t$, where F_i are faces of K. In the second subdivision in each such face we pick some point close to the point labeled F_0. It is important to note, that since all the point choices were made up to a small perturbation, we can always achieve that:

(1) all the d-simplices of $\operatorname{bd}^2 K$ which contain points from L have only new vertices, since all the other d-simplices can be made arbitrarily small, see Fig. 5.2;
(2) for every d-simplex ρ of $\operatorname{bd}^2 K$ the points from $\rho \cap L$ are in general position in ρ.

This leads to the following corollary.

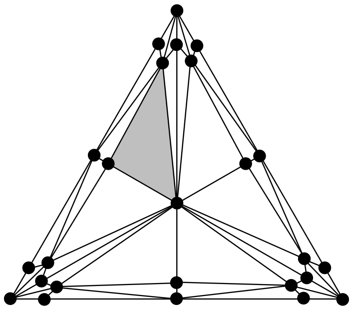

Fig. 5.2. The second barycentric subdivision used in the proof of Corollary 5.3. The darker triangle is an example of a d-simplex which may contain points from L.

Corollary 5.3. *Assume* $\dim L = 0$, $K = \Delta^d$, *and* $|L| \subset \operatorname{int} K$, *then there exists an iterated barycentric subdivision* $\operatorname{bd}^t K$, *where* $t = \lceil \log_{(d+1)!} n \rceil + 3$, *such that L is a subcomplex of* $\operatorname{bd}^t K$.

Proof. The only d-simplices of $\operatorname{bd}^2 K$ which may contain points from L will be of the form

$$\big(([d+1]), ([d+1] \supset F_d), \dots, ([d+1] \supset F_d \supset F_{d-1} \supset \cdots \supset F_1)\big). \tag{5.2}$$

Any d-simplex will be of the form $(A_1, \dots, A_{d+1})$, where each A_i is an inclusion sequence of i faces of K, and the sequences are included in each other. If this d-simplex is not of the form (5.2), then there will exist A_i and A_j with $i \neq j$, but with the same minimal set in the inclusion. By our construction, these two vertices will be arbitrarily close to each other, making the simplex arbitrarily small; call such a simplex degenerate. We can then make sure all the degenerate simplices containing A_{d+1} do not contain any points from L by taking the corresponding point of the second barycentric subdivision sufficiently close to the vertex $\min A_{d+1}$. $\square$

Finally, let us drop the condition that K is a d-simplex. In this situation the points of L are distributed over several simplices of K. Still, we can apply the same procedure as before: produce the first barycentric subdivision by placing new points close to the actual barycenters, and then produce the second barycentric subdivision by putting a new point close to F_0 for every simplex indexed with $F_0 \subset F_1 \subset \cdots \subset F_t$.

Theorem 5.4. *Assume* $\dim L = 0$, $\dim K = m$, $|L| \subseteq |K|$, *and for every simplex* $\sigma \in K$ *the points* $L \cap \operatorname{int} \sigma$ *are in general position in* σ. *For every* $i = 0, \dots, m$, *let* $n_i = \max_\sigma |L \cap \operatorname{int} \sigma|$, *where the maximum is taken over all i-simplices* $\sigma \in K$, *and set* $t := 1 + \max_{i=0,\dots,m} \lceil \log_{(i+1)!} n_i \rceil$. *Then, there exists an iterated barycentric subdivision* $\operatorname{bd}^t K$, *such that L is a subcomplex of* $\operatorname{bd}^t K$. *If the general position condition is dropped, then there exists an iterated barycentric subdivision* $\operatorname{bd}^{t+2} K$, *such that L is a subcomplex of* $\operatorname{bd}^{t+2} K$.

Proof. The proof is essentially the same as that of Corollary 5.3. For $I = \{i_1, \dots, i_p\}$, we set $\mathcal{F}_I := F_{i_1} \subset \cdots \subset F_{i_p}$. The q-simplices are now indexed by the q-tuples $(\mathcal{F}_{I_1}, \dots, \mathcal{F}_{I_q})$, where $I_1 \subset \cdots \subset I_q$. The volume of the q-simplex will not be arbitrarily small if in addition the minima $\min I_1, \dots, \min I_q$ are all different. Hence the top-dimensional nondegenerate simplices will be of the form $((F_t), (F_t \supset F_{t-1}), \dots, (F_t \supset \cdots \supset F_1))$. $\square$

5.3. Application to sensor networks

Now we would like to apply our results in the context of sensor networks. Consider the following model. We have a certain finite number of sensors in $\mathbb{R}^d$ which are modeled as points. In addition we have some other sensors, which we call *master sensors*; there should be quite a bit fewer of these, and they are supposed to help to cluster our sensors in some balanced way. We want to be able to maintain a balanced system of master sensors without introducing too many of these.

Specifically, the original and the master sensors are points in $\mathbb{R}^d$. The master sensors are vertices of a geometric simplicial complex K which is the allowed locus of the sensors; without loss of generality, we may choose K to be just a simplex. Over time, more sensors are added to the network, and we are asked to maintain the system of master sensors so that every d-simplex in the corresponding triangulation of K has at most r sensors. Small perturbations of sensors are allowed.

One can use our results on stellar equipartitions to approach this model. Whenever a number of sensors in a specific d-simplex Δ reaches $r+1$, we simply use our Corollary 3.6 to locate in $O(r^2)$ time the center a of the stellar subdivision, where all the sensors lie in the interiors of the d-simplices of $\operatorname{sd}(a, \Delta)$, and each d-simplex contains either $\lfloor \frac{r+1}{d+1} \rfloor$ or $\lceil \frac{r+1}{d+1} \rceil$ sensors. If the sensors are not in general position we simply perturb them slightly, so that Corollary 3.6 can be applied.

To get a clean bound on the number of master sensors, assume K is just a d-simplex and we started with $d+1$ master sensors – the vertices of K; the generalization to an arbitrary K is straightforward. As a result of our subdivision procedure above, the number of sensors in each d-simplex in the end is between $\lfloor \frac{r+1}{d+1} \rfloor$ and r. Let m be the number of master sensors which we introduced, then $t := 1 + md$ is the number of d-simplices in the final subdivision (each new master sensor increases the number of d-simplices by d). Denoting by n the number of sensors, we obtain the bound

$$tr \geqslant n \geqslant t \left\lfloor \frac{r+1}{d+1} \right\rfloor,$$

and hence we get the following bound for the number of master sensors

$$\frac{n/\lfloor (r+1)/(d+1) \rfloor - 1}{d} \geqslant m \geqslant \frac{n-r}{rd},$$

which implies weaker but simpler bound

$$\frac{n(d+1)}{(r-d)d} \geqslant m \geqslant \frac{n-r}{rd}.$$

6. Final remarks and open questions

In this section we summarize the open questions, some of which have already appeared scattered throughout the text.

Open Problem 6.1.

(a) Given 6^t points in a triangle, how many iterated barycentric subdivisions will it take in the worst case to incorporate all the points as vertices?
(b) The same question for $((d+1)!)^t$ points in a d-simplex.
(c) The same question for n points in a d-simplex.
(d) The same questions in the tropical setting.

The results of Section 5.2 show that the answer in (a) and in (b) is less than or equal to $t+3$. For (a), notice that the t-th barycentric subdivision has $1 + \frac{3}{2} \cdot 2^t + \frac{1}{2} \cdot 6^t$ vertices, from which $1 - \frac{3}{2} \cdot 2^t + \frac{1}{2} \cdot 6^t$ are internal ones. For (b), we notice that one can show that the t-th barycentric subdivision has fewer than $((d+1)!)^t$ vertices. This implies that the answer in both cases is somewhere between $t+1$ and $t+3$. In fact, we believe it is the same answer.

Open Problem 6.2. Compute $\eta(\Delta^d, L)$, where L is the chromatic subdivision of the d-simplex, see Section 1.

Open Problem 6.3. Prove the tropical analog of Theorem 4.2.

This might be not too hard. Unfortunately, our proof of the simplicial case uses volume estimates which do not generalize in a straightforward way to the tropical (infinite) case.

Open Problem 6.4. Give a more combinatorial description of the stellar discriminant and understand its topological properties in the spirit of arrangement theory.

It would be nice to have a presentation as a union of simple pieces, and it would be nice to have some combinatorial control as to how these pieces intersect.

Open Problem 6.5. Describe the locus of point configurations for which the stellar equipartition exists.

The last problem is probably quite hard and is meant to be sort of a guiding beacon. Further conjectures can be found in [8].

Acknowledgements

The author would like to thank the Mathematical Sciences Research Institute at Berkeley, Stanford University, as well as University of Buenos Aires for the hospitality during the time this research was done. He would also like to thank the anonymous referee for finding a mistake in the proof of one of the statements in the original submission, as well as for providing several other valuable comments.

References

[1] E.M. Feichtner, De Concini–Procesi wonderful arrangement models: a discrete geometer's point of view, in: Combinatorial and Computational Geometry, in: Math. Sci. Res. Inst. Publ., vol. 52, Cambridge Univ. Press, Cambridge, 2005, pp. 333–360.
[2] E.M. Feichtner, D.N. Kozlov, Incidence combinatorics of resolutions, Selecta Math. (N.S.) 10 (1) (2004) 37–60.
[3] E. Gafni, personal communication, 2010.
[4] L.C. Glaser, Geometrical Combinatorial Topology, vol. I, Van Nostrand Reinhold, New York, 1970.
[5] M.P. Herlihy, N. Shavit, The topological structure of asynchronous computability, J. ACM (November 1999) (preliminary versions in STOC'93 and STOC'94).
[6] I. Itenberg, G. Mikhalkin, E. Shustin, Tropical Algebraic Geometry, second edition, Oberwolfach Semin., vol. 35, Birkhäuser Verlag, Basel, 2009.
[7] D.N. Kozlov, Combinatorial Algebraic Topology, Algorithms Comput. Math., vol. 21, Springer-Verlag, Berlin, Heidelberg, 2008, XX, 390 pp., 115 illus.
[8] D.N. Kozlov, Some conjectures concerning complexity of PL subdivisions, Electron. Notes Theor. Comput. Sci. 283 (2012) 153–157.
[9] P. Orlik, H. Terao, Arrangements of Hyperplanes, Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 300, Springer-Verlag, Berlin, 1992.
[10] J. Richter-Gebert, B. Sturmfels, T. Theobald, First steps in tropical geometry, in: Idempotent Mathematics and Mathematical Physics, in: Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 289–317.